

# Dilatonic effects near naked singularities

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## Abstract

Static spherically symmetric solutions of 4d Brans-Dicke theory include a set of naked singularity solutions. Dilatonic effects near the naked singularities result in either a shielding or an antishielding effect from intruding massive test particles. One result is that for a portion of the solution parameter space, no communication between the singularity and a distant observer is possible via massive particle exchanges. Kaluza-Klein gravity is considered as a special case.

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## I. INTRODUCTION

The static spherically symmetric vacuum solutions (see, for example, [1–3]) of Brans-Dicke theory[4] include the Schwarzschild solution, along with a set of solutions describing naked singularities. The solutions of the 4-dimensional Brans-Dicke theory were first constructed in the Jordan-Brans-Dicke frame by Brans[1]. However, a conformal transformation to the Einstein frame transforms the matter-free Brans-Dicke theory into an Einstein theory of gravity with a minimally coupled massless scalar field. The higher dimensional spherically symmetric vacuum solutions of this theory in  $D \geq 4$  dimensions were provided by Xanthopoulos and Zannias[2], and include the 4-dimensional Brans solutions, expressed in the Einstein frame. These two parameter  $D$ -dimensional solutions were shown to correspond to naked singularities for all cases except for the case corresponding to the  $D$ -dimensional Schwarzschild solution.

Cai and Myung[3] have also studied such solutions for the case of  $D \geq 4$  dimensional Brans-Dicke theory in both the Jordan frame and the Einstein frame, as well as examining solutions for the  $D$ -dimensional Brans-Dicke-Maxwell theory. Again, for the case of neutral nonrotating solutions the only “black hole” solution is the Schwarzschild solution with a constant dilaton, and the rest of the solutions, describing naked singularities, have attendant nontrivial dilaton fields. A 4d dilaton field  $\tilde{\phi}(x^\mu)$  in the Brans-Dicke theory, and its spherically symmetric static solutions, can be connected in a straightforward way with a scale factor  $b(x^\mu)$  of isotropically, toroidally compactified extra spatial dimensions in a higher dimensional theory of Einstein gravity. Specifically,  $(4 + n)$ -dimensional pure Einstein gravity, when dimensionally reduced to 4 dimensions, can take the form of a 4d Brans-Dicke theory, with a Brans-Dicke parameter  $\omega_{BD}$  that depends upon the number  $n$  of compactified dimensions and a Brans-Dicke field  $\tilde{\phi}$  that is related to the scale factor  $b$  of the  $n$  extra dimensions. In this case of Kaluza-Klein gravity, a variation in the dilaton field  $\tilde{\phi}(x)$  is associated with a variation in the extra dimensional scale factor  $b(x)$ . In the Einstein frame of the 4d action which includes matter, there is a dilaton coupling to matter which results in particle masses that depend on  $b(x)$ .

In [5] a study was made of the reflection and transmission of massless and massive particles through a “wall” of varying  $\tilde{\phi}$  or  $b$  in the Einstein frame of a Kaluza-Klein theory with  $n$  extra dimensions that are isotropically compactified. This study assumed a flat 4d spacetime background, and the qualitative nature of the results agree with those of previous studies with nondilatonic walls[6–9]. However, caution must be exercised in applying these results to regions of strong spacetime curvature, since there are gravitational redshifting effects to consider that may modify the energy dependent reflection coefficient  $\mathcal{R}(\omega)$  in such regions.

Problems associated with a calculation of reflection and transmission coefficients of massive particles in strong gravitational fields are sidetracked here through an approach that focuses upon the kinematics of massive particles propagating in a dilaton-gravity background of the type described by the neutral static spherically symmetric solutions studied by Brans[1], Xanthopoulos and Zannias[2], and Cai and Myung[3]. This approach allows an inference of some basic physical effects on massive particles of the “dilaton cloud” surrounding the Brans-Dicke naked singularities. This dilaton cloud is described by the exact analytical solution for the Brans-Dicke scalar field.

Therefore, the approach presented here simply relies on kinematical properties of massive test particles propagating in a dilatonic field of a neutral nonrotating naked singularity, considered in the Einstein frame. Regions where a particle is kinematically allowed to propagate are defined, and some general comments are offered concerning purely radial motion. (Also see ref.[10], where geodesic motion was considered in the Jordan frame, using nonisotropic coordinates.) One set of solutions is associated with an attractive dilatonic force on test particles, and another set of solutions exerts a repulsive dilatonic force on test particles.

The results that are obtained suggest that, for a portion of the solution parameter space describing naked singularities surrounded by a repulsive dilaton cloud, test particles with nonzero mass infalling from radial infinity, regardless of the particle energy, can not reach the singularity and are reflected back out. That is, for a range of solution parameters, certain regions close to the singularity become kinematically forbidden. Consequently, for these solutions no communication between the singularity and a distant observer via massive particle exchanges is possible. (This situation does not appear to have been pointed out in ref.[10].) These results are also expressed in terms of the number  $n$  of compactified extra dimensions for the special case of Kaluza-Klein gravity.

## II. JORDAN AND EINSTEIN FRAME REPRESENTATIONS

### A. Brans-Dicke gravity in 4d

The action for four dimensional Brans-Dicke theory is given by

$$S = \frac{1}{16\pi} \int d^4x \sqrt{\tilde{g}} \left\{ \tilde{\phi} \tilde{R} + \frac{\omega_{BD}}{\tilde{\phi}} \tilde{g}^{\mu\nu} \partial_\mu \tilde{\phi} \partial_\nu \tilde{\phi} \right\} + S_m(\tilde{g}_{\mu\nu}) \quad (2.1)$$

where  $S_m$  is the matter action, Newton’s constant is set to unity,  $G = 1$ , and  $\tilde{g} = |\det(\tilde{g}_{\mu\nu})|$ . This action is expressed in the Jordan-Brans-Dicke conformal frame, or just the Jordan frame, for short. The metric in this frame is  $\tilde{g}_{\mu\nu}$  and the scalar curvature  $\tilde{R}$  is built from this

metric. (A metric with signature  $(+, -, -, -)$  is used.) The dimensionless constant  $\omega_{BD}$  is the Brans-Dicke parameter, and the matter action  $S_m(\tilde{g}_{\mu\nu})$  is constructed using the Jordan frame metric  $\tilde{g}_{\mu\nu}$ . For a test particle of constant mass  $m_0$  in the Jordan frame,  $S \propto \int m_0 d\tilde{s}$ , where  $d\tilde{s} = \sqrt{\tilde{g}_{\mu\nu} dx^\mu dx^\nu}$ .

A conformal transformation to the Einstein frame is given by[3]

$$g_{\mu\nu} = \tilde{\phi} \tilde{g}_{\mu\nu}, \quad g^{\mu\nu} = \tilde{\phi}^{-1} \tilde{g}^{\mu\nu}, \quad \sqrt{g} = \tilde{\phi}^2 \sqrt{\tilde{g}}, \quad \phi = \sqrt{2a} \ln \tilde{\phi}, \quad a = \omega_{BD} + \frac{3}{2} \quad (2.2)$$

and the action in the Einstein frame then takes the form

$$S = \frac{1}{16\pi} \int d^4x \sqrt{g} \left\{ R + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right\} + S_m(\tilde{\phi}^{-1} g_{\mu\nu}) \quad (2.3)$$

where  $R$  is built from  $g_{\mu\nu}$  and Einstein gravity is coupled to a massless scalar dilaton field  $\phi$ . Note, however, that from the Einstein frame perspective, mass becomes position dependent in general, since we can write

$$S_m \propto \int m_0 d\tilde{s} = \int m_0 \tilde{\phi}^{-1/2} ds = \int m ds \quad (2.4)$$

where  $d\tilde{s} = \tilde{\phi}^{-1/2} ds$  and  $m$  is the mass in the Einstein frame, given in terms of the dilaton field  $\tilde{\phi}$  by

$$m = \tilde{\phi}^{-1/2} m_0 \quad (2.5)$$

Therefore, a particle having a constant mass  $m_0$  in the Jordan frame will have a mass  $m = \tilde{\phi}^{-1/2} m_0$  in the Einstein frame[11].

## B. Kaluza-Klein gravity

Consider an action describing pure Einstein gravity coupled to matter in  $D = 4 + n$  dimensions,

$$S_D = \int d^Dx \sqrt{\tilde{g}_D} \left\{ \frac{1}{2\kappa_D^2} \left[ \tilde{R}_D[\tilde{g}_{MN}] - 2\Lambda \right] + \tilde{\mathcal{L}}_D \right\} \quad (2.6)$$

in a spacetime described by

$$d\tilde{s}_D^2 = \tilde{g}_{MN} dx^M dx^N = \tilde{g}_{\mu\nu}(x) dx^\mu dx^\nu - b^2(x) \gamma_{mn}(y) dy^m dy^n \quad (2.7)$$

with  $x^M = (x^\mu, y^m)$ ,  $M, N = 0, 1, 2, 3, \dots, D-1$ , and  $\tilde{g}_D = |\det \tilde{g}_{MN}|$ . There are  $n$  extra compact space coordinates labeled by  $y^m$ , and the metric for the extra dimensions is

$\tilde{g}_{mn}(x, y) = -b^2(x)\gamma_{mn}(y)$  with  $\gamma_{mn}(y)$  describing the geometry of the compact space, and the isotropic extra dimensional scale factor  $b(x^\mu)$  is assumed to be  $y$  independent. (The metric has signature  $(+, -, \dots, -)$ .) The factor  $\kappa_D^2 = 8\pi G_D$  for the  $D$  dimensional spacetime is related to the corresponding 4d one by  $\kappa_D^2 = 8\pi G_D = V_y \kappa^2 = V_y(8\pi G)$  where  $V_y$  is the coordinate volume of the extra dimensional space and  $\kappa^2 = 8\pi G = 8\pi$  ( $G = 1$ ) is the inverse of the 4d reduced Planck mass.  $\tilde{\mathcal{L}}_D$  and  $\Lambda$  are the matter Lagrangian and cosmological constant, respectively, in the  $D$  dimensional space.

Borrowing results used in [12] and [5] to express the action as an effective 4d action, an integration over the  $y$  coordinates leaves (sign errors appearing in eq.(7) of ref.[5], having no effect there, are corrected here)

$$S = \int d^4x \sqrt{\tilde{g}} \left\{ \frac{1}{2\kappa^2} [b^n \tilde{R}[\tilde{g}_{\mu\nu}] + 2nb^{n-1} \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\nabla}_\nu b + n(n-1)b^{n-2} \tilde{g}^{\mu\nu} (\tilde{\nabla}_\mu b)(\tilde{\nabla}_\nu b) + n(n-1)kb^{n-2}] + b^n \left[ \mathcal{L}_D - \frac{\Lambda}{\kappa^2} \right] \right\} \quad (2.8)$$

where  $\mathcal{L}_D = V_y \tilde{\mathcal{L}}_D$  is a normalized Lagrangian and  $\tilde{R}[\tilde{g}_{\mu\nu}]$  is the Ricci scalar built from  $\tilde{g}_{\mu\nu}$ , etc. (See [12] and [5] for details.) The metric  $\tilde{g}_{\mu\nu}$  then acts as a 4d Jordan frame metric. The constant  $n(n-1)k = \tilde{R}[\gamma_{mn}]$  gives the curvature of the internal space, which we will set equal to zero. Dropping a total divergence, this action can be rewritten as

$$S = \int d^4x \sqrt{\tilde{g}} \left\{ \frac{1}{2\kappa^2} [b^n \tilde{R}[\tilde{g}_{\mu\nu}] - n(n-1)b^{n-2} \tilde{g}^{\mu\nu} (\tilde{\nabla}_\mu b)(\tilde{\nabla}_\nu b) + n(n-1)kb^{n-2}] + b^n \left[ \mathcal{L}_D - \frac{\Lambda}{\kappa^2} \right] \right\} \quad (2.9)$$

Upon setting  $\tilde{R}[\gamma_{mn}] = n(n-1)k = 0$  (toroidal compactification, for example) and  $\Lambda = 0$ , this assumes the form of (2.1) with the identifications

$$\tilde{\phi} = b^n, \quad \omega_{BD} = -1 + \frac{1}{n}, \quad \tilde{\mathcal{L}}_m = b^n \mathcal{L}_D \quad (2.10)$$

Use of the conformal transformation of (2.2) allows the action to be represented in the

Einstein frame, as given by (2.3), with  $\phi = \sqrt{2a} \ln \tilde{\phi}$ ,  $a = \omega_{BD} + \frac{3}{2}$  (which gives  $a = \frac{n+2}{2n}$  for the Kaluza-Klein case), and an effective 4d matter Lagrangian density  $\mathcal{L}_m = b^{-n} \mathcal{L}_D$ .

The 4d Einstein frame spacetime is described by  $ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$  and is related to the 4d Jordan frame spacetime of  $d\tilde{s}^2 = \tilde{g}_{\mu\nu}(x)dx^\mu dx^\nu$  by the conformal transformation  $\tilde{g}_{\mu\nu} = \tilde{\phi}^{-1} g_{\mu\nu}$ . The two line elements are related by  $ds^2 = \tilde{\phi} d\tilde{s}^2$ . We take the classical action for a test particle to be  $S \propto \int m_0 d\tilde{s}_D$  and assume the particle worldline to be independent

of the coordinates  $y^m$ , so that  $dy^m = 0$  along the worldline. Then along the worldline  $d\tilde{s}_D = d\tilde{s} = \tilde{\phi}^{-1/2}ds$  so that  $S \propto \int m_0 d\tilde{s}_D = \int m_0 d\tilde{s} = \int m ds$ , where

$$m = \tilde{\phi}^{-1/2}m_0 = b^{-\frac{n}{2}}m_0 \quad (2.11)$$

In the Jordan frame particles have constant,  $\tilde{\phi}$  independent masses and follow geodesics, whereas in the Einstein frame masses become  $\tilde{\phi}$  dependent and paths are generally not geodesics due to the  $x^\mu$  dependence of the particle mass[11]. The 4d matter fields in the field theoretic Lagrangian density  $\mathcal{L}_m$  can be rescaled in the Einstein frame, but masses pick up a  $\tilde{\phi}$  dependence (therefore a  $b$  dependence for the Kaluza-Klein case).

### III. SPHERICALLY SYMMETRIC VACUUM SOLUTIONS

The vacuum solutions of the field equations obtained from the Einstein frame action of (2.3), as well as higher dimensional generalizations, were obtained by Xanthopoulos and Zannias[2]. These were also presented by Cai and Myung[3]. The static neutral solutions, with isotropic coordinates, are presented here for the special 4d case:

$$ds^2 = e^f dt^2 - e^{-h}(dr^2 + r^2 d\Omega^2) \quad (3.1)$$

$$e^f = g_{00} = \xi^{2\gamma}; \quad \xi = \left( \frac{r - r_0}{r + r_0} \right) \quad (3.2)$$

$$e^{-h} = |g_{rr}| = \left( 1 - \frac{r_0^2}{r^2} \right)^2 \xi^{-2\gamma} = e^{-f} \left( 1 - \frac{r_0^2}{r^2} \right)^2 \quad (3.3)$$

$$\phi = \pm \tilde{\gamma} \ln \xi = \sqrt{2a} \ln \tilde{\phi}; \quad (3.4a)$$

$$\tilde{\phi} = \xi^\Gamma \quad (3.4b)$$

where  $r_0$  and  $\gamma$  are integration constants ( $r_0 > 0$ ), and we have defined

$$\xi = \left( \frac{r - r_0}{r + r_0} \right) \leq 1, \quad \tilde{\gamma} = [4(1 - \gamma^2)]^{1/2}, \quad \Gamma = \pm \frac{\tilde{\gamma}}{\sqrt{2a}} = \pm \left[ \frac{2}{a}(1 - \gamma^2) \right]^{1/2} \quad (3.5)$$

These are the Einstein frame fields and solutions, with  $0 \leq \gamma \leq 1$  for the description of physical (nonnegative ADM mass) solutions. There is a naked singularity at  $r = r_0$  where  $R = g^{\mu\nu} R_{\mu\nu} \rightarrow \infty$  unless  $\gamma = 1$  and  $\phi = 0$  (the Schwarzschild solution).

*Note:* In the set of vacuum solutions presented in ref.[2], only the solution with the + sign in (3.4a), i.e.,  $\phi = +\tilde{\gamma} \ln \xi$ , is presented. However, the second solution  $\phi = -\tilde{\gamma} \ln \xi$  is seen to exist due to the invariance of the action and equations of motion (EoM) under the transformations  $g_{\mu\nu} \rightarrow g_{\mu\nu}$ ,  $\phi \rightarrow -\phi$ . Thus if  $\phi$  is a solution to the EoM, then so is  $-\phi$  (see, for example, refs.[3] and[13]). Therefore  $\phi$  can be positive or negative, and the Brans-Dicke scalar  $\tilde{\phi} = \xi^\Gamma = \xi^{\pm|\Gamma|}$  can be either a decreasing or an increasing function of  $r$  and  $\xi$ .

For the case of Kaluza-Klein gravity in 4d, (2.10) gives  $\tilde{\phi} = b^n$  so that  $b = \xi^{\Gamma/n}$  with  $\omega_{BD} = (1 - n)/n$  and  $a = (n + 2)/2n$ . In this case  $\Gamma$  takes a value

$$\Gamma = \pm 2 \left[ \left( \frac{n}{n+2} \right) (1 - \gamma^2) \right]^{1/2} \quad (3.6)$$

This allows the scale factor to either shrink to zero or to blow up as  $r \rightarrow r_0$ ,

$$b \rightarrow \begin{cases} 0, & \Gamma > 0 \\ \infty, & \Gamma < 0 \\ 1, & \Gamma = 0 \end{cases} \quad \text{as } r \rightarrow r_0 \quad (3.7)$$

These possibilities were pointed out by Davidson and Owen[14] for the case of one extra dimension,  $n = 1$ . The case  $\gamma = 1$ ,  $\Gamma = 0$  corresponds to the Schwarzschild solution, with  $\phi = 0$ ,  $\tilde{\phi} = b = 1$ .

#### IV. KINEMATICAL CONSTRAINT FOR MASSIVE PARTICLE

Now attention is focused on a spinless test particle of arbitrary nonzero mass  $m(r)$  propagating in the spacetime of (3.1). The particle mass  $m$  is generally position dependent, due to the dilaton field,  $m = \tilde{\phi}^{-1/2} m_0$ . We will obtain kinematical constraints on the allowed and forbidden regions of particle propagation by considering a classical “geodesic” approach.

##### A. Geodesic Approach

Consider now a neutral test particle moving under the influence of the gravitational and dilatonic fields present in the Einstein frame. In this frame, again,  $m(r)$  varies with radial position  $r$  of the particle. This gives rise to a dilatonic force correction to the geodesic equation[11] in the Einstein frame,

$$\frac{d}{ds} (m g_{\mu\nu} u^\nu) - \frac{1}{2} m (\partial_\mu g_{\alpha\beta}) u^\alpha u^\beta - \partial_\mu m = 0 \quad (4.1)$$

where  $u^\mu = dx^\mu/ds$  and the  $\partial_\mu m$  term arises due to the variability of  $m$ . The velocities  $u^\mu$  satisfy the constraint  $g_{\mu\nu}u^\mu u^\nu = 1$ . Focusing upon a particle with a purely radial trajectory with  $\theta, \phi$  held constant, we have the components  $u^0 = u^t$  and  $u^1 = u^r$  being nonvanishing, in general. The vacuum solutions are described by a time independent, diagonal metric and dilaton,  $\partial_0 g_{\mu\nu} = 0$ ,  $\partial_0 m = 0$ , and the geodesic equations for  $u^0$  and  $u^r$  are given by

$$\frac{d}{ds}(m u_0) - \frac{1}{2}m(\partial_0 g_{\alpha\beta})u^\alpha u^\beta - \partial_0 m = 0 \quad (4.2)$$

$$\frac{d}{ds}(m u_r) - \frac{1}{2}m(\partial_r g_{\alpha\beta})u^\alpha u^\beta - \partial_r m = 0 \quad (4.3)$$

These reduce to

$$\frac{d}{ds}(m u_0) = \frac{d}{ds}(m g_{00} u^0) = 0 \quad (4.4)$$

$$\frac{d}{ds}(m g_{rr} u^r) - \frac{1}{2}m [(\partial_r g_{00})(u^0)^2 + (\partial_r g_{rr})(u^r)^2] - \partial_r m = 0 \quad (4.5)$$

The first equation gives

$$u^0 = \frac{\alpha}{m g_{00}} \quad (4.6)$$

where  $\alpha = m u_0 = p_0$  is a constant. The constraint  $u_\mu u^\mu = g_{00}(u^0)^2 + g_{rr}(u^r)^2 = 1$  then produces

$$(u^r)^2 = \left(\frac{dr}{ds}\right)^2 = \frac{1}{g_{rr}} \left[1 - \frac{\alpha^2}{m^2 g_{00}}\right] \quad (4.7)$$

For  $r > r_0$  where  $g_{rr} < 0$  is finite, the above constraint requires that

$$m^2 g_{00} = m_0^2 \tilde{\phi}^{-1} g_{00} \leq \alpha^2 \quad (4.8)$$

in regions where the particle is kinematically allowed to propagate, with  $(u^r)^2 \geq 0$ . Turning points of the radial motion are located where  $(u^r)^2 = 0$ .

Asymptotically, we have  $g_{00} \rightarrow 1$ ,  $g_{rr} \rightarrow -1$ , and therefore at  $r = \infty$ ,  $u_\mu u^\mu = (u_\infty^0)^2 - (u_\infty^r)^2 = 1$ . The proper time of the particle moving with radial speed  $v$  at  $r = \infty$  is given by  $ds = d\tau = \gamma_{rel}^{-1} dt$ , so that  $u^0 = dt/ds = \gamma_{rel} = 1/\sqrt{1-v^2}$ . (Here  $\gamma_{rel}$  is the ordinary special relativistic gamma factor, not to be confused with the solution parameter  $\gamma$ .) Therefore

$$(u_\infty^r)^2 = \gamma_{rel}^2 - 1 \quad (4.9)$$

Now using the definition of  $\alpha$ ,

$$\alpha = m g_{00} u^0 = (m g_{00} u^0)|_\infty = \gamma_{rel} m_0 = E_0 = \omega_0 \quad (4.10)$$

With this identification of  $\alpha = \omega_0$  the kinematical constraint in (4.8) becomes  $(m^2/\omega_0^2) g_{00} \leq 1$ , or

$$\frac{m^2}{\omega^2} = \frac{m_0^2}{\omega_0^2} (\tilde{\phi}^{-1} g_{00}) \leq 1 \quad (\text{kinematically allowed, } \omega = \frac{\omega_0}{\sqrt{g_{00}}}) \quad (4.11)$$



## B. Radial motion

We can define an energy parameter  $\mathcal{E} = E_0/m_0 = \omega_0/m_0$ , which is the asymptotic energy per unit mass value. Eq. (4.7) can then be expressed as

$$(u^r)^2 = \left(\frac{dr}{ds}\right)^2 = \frac{1}{|g_{rr}|} \left[ \mathcal{E}^2 \frac{\tilde{\phi}}{g_{00}} - 1 \right] \quad (4.12)$$

The kinematically allowed region where the particle can propagate can then be written as

$$\mathcal{E}^2 \frac{\tilde{\phi}}{g_{00}} = \mathcal{E}^2 \left( \frac{r - r_0}{r + r_0} \right)^{-(2\gamma - \Gamma)} = \mathcal{E}^2 \xi^{-(2\gamma - \Gamma)} \geq 1 \quad (\text{kinematically allowed}) \quad (4.13)$$

Turning points are located where  $(u^r)^2 = 0$ , or  $\mathcal{E}^2 \frac{\tilde{\phi}}{g_{00}} = \mathcal{E}^2 \xi^{-(2\gamma - \Gamma)} = 1$ . A particle can escape to  $r \rightarrow \infty$  ( $\xi \rightarrow 1$ ) if  $\mathcal{E} > 1$ . However, if  $2\gamma - \Gamma \geq 0$  but  $\mathcal{E} < 1$ ,  $\xi = 1$  is not kinematically allowed, and the particle will be gravitationally trapped. For  $\mathcal{E} > 1$  and  $2\gamma - \Gamma \geq 0$ , there are no turning points. In this case a particle can plunge inward from radial infinity all the way to the singularity, and a particle ejected from the singularity region can escape to infinity. Note that the condition  $2\gamma - \Gamma \geq 0$  is obtained for all  $\Gamma \leq 0$  and for positive values of  $\Gamma$  for which  $\Gamma \leq 2\gamma$ .

If, on the other hand,  $2\gamma - \Gamma < 0$ , then (4.13) implies that  $1 > \xi^{|2\gamma - \Gamma|} \geq 1/\mathcal{E}^2$ , and since  $\xi < 1$  we must require  $\mathcal{E} > 1$ , and the particle cannot be gravitationally trapped in this case, and we have  $\xi \in [\xi_{\min}, 1)$  and  $r \in [r_{\min}, \infty)$ , where  $\xi_{\min} = \mathcal{E}^{-2/|2\gamma - \Gamma|}$  locates the turning point where  $r = r_{\min}$  and  $u^r = 0$ . The condition  $2\gamma - \Gamma < 0$  requires positive values of  $\Gamma$ , with  $\Gamma > 2\gamma$ .

## V. KINEMATICALLY ALLOWED AND FORBIDDEN REGIONS

The constraint (4.11) or (4.13) for kinematically allowed regions is  $m^2/\omega^2 \leq 1$  and the kinematically forbidden region has  $m^2/\omega^2 > 1$ . The condition (4.13) for kinematically allowed regions is

$$\frac{m^2}{\omega^2} = \frac{m_0^2}{\omega_0^2} (\tilde{\phi}^{-1} g_{00}) = \frac{m_0^2}{\omega_0^2} \left( \frac{r - r_0}{r + r_0} \right)^{2\gamma - \Gamma} = \frac{\xi^{2\gamma - \Gamma}}{\mathcal{E}^2} \leq 1 \quad (\text{kinematically allowed}) \quad (5.1)$$

As  $r \rightarrow r_0$  ( $\xi \rightarrow 0$ ),

$$\frac{m^2}{\omega^2} \rightarrow \begin{cases} 0, & \text{if } 2\gamma - \Gamma > 0 : r \rightarrow r_0 \text{ is allowed, } 2\gamma > \Gamma \\ \infty, & \text{if } 2\gamma - \Gamma < 0 : r \rightarrow r_0 \text{ is forbidden, } 2\gamma < \Gamma \\ \mathcal{E}^{-2}, & \text{if } 2\gamma - \Gamma = 0 : r \rightarrow r_0 \text{ is allowed, } 2\gamma = \Gamma \end{cases} \quad (5.2)$$

The particle can travel all the way to  $r_0$  with  $m/\omega \leq 1$  if  $2\gamma - \Gamma \geq 0$ , but it cannot reach  $r = r_0$  for  $2\gamma - \Gamma < 0$ , i.e. if  $2\gamma < \Gamma$ . For the case  $2\gamma = \Gamma$ , we must have  $\mathcal{E} \geq 1$ .

Now, for the case  $\Gamma > 2\gamma \geq 0$  we have  $\frac{2\gamma}{\Gamma} \geq 0$ . Then the singularity  $r = r_0$  is forbidden only for  $\frac{2\gamma}{\Gamma} < 1$ , and this leads to the parameter constraint  $\gamma = \sqrt{\frac{(2\gamma/\Gamma)^2}{(2\gamma/\Gamma)^2 + 2a}} < \frac{1}{\sqrt{1+2a}}$ : i.e., for

$$\Gamma > 0, \quad \gamma < \frac{1}{\sqrt{1+2a}} : \quad r = r_0 \text{ is forbidden} \quad (5.3)$$

The *singularity is a forbidden region* for a vacuum solution with  $\Gamma > 0$  and  $\gamma < (1+2a)^{-1/2}$ . For the case of Kaluza-Klein gravity,  $a = (n+2)/2n$  and (5.3) translates into

$$\Gamma > 0, \quad \gamma < \frac{1}{\sqrt{2}} \sqrt{\frac{n}{n+1}} : \quad r = r_0 \text{ is forbidden} \quad (5.4)$$

The naked singularity solutions with positive  $\Gamma$  and  $\gamma < (1+2a)^{-1/2}$  do not allow massive particles to reach the singular point  $r_0$ , but the other solutions with  $\gamma > (1+2a)^{-1/2}$  do. For the case of Kaluza-Klein gravity, setting  $n = 1$  corresponds to  $\gamma < \frac{1}{2}$ , while for large  $n$  this approaches  $\gamma < \frac{1}{\sqrt{2}} = .707$ . In either case, for Kaluza-Klein gravity there is a sizable portion of the  $\gamma \in [0, 1]$  parameter space for which the naked singularity solutions do not allow massive particles to propagate in the immediate vicinity of the singularity.

On the other hand, in the context of a Brans-Dicke theory with a massless scalar field  $\tilde{\phi}$ , which is subject to the constraint that  $\omega_{BD} \gg 1$  ( $a \gg 1$ ), there is only a small portion of the  $\gamma$  parameter space for which the singularity is untouchable by massive particles. For example, the solar system constraint on  $\omega_{BD}$  for a *massless* Brans-Dicke theory requires[15]  $\omega_{BD} > 40,000$ . In this case, for  $\omega_{BD} \sim a \gtrsim 4 \times 10^4$ , (3.4b) implies that  $|\Gamma| \lesssim 10^{-2}$ , which lies very close to the Schwarzschild limit  $\Gamma = 0$ , and hence  $\tilde{\phi} = \xi^\Gamma$  is very slowly varying, and may make the  $\gamma \neq 1$  Brans-Dicke solutions  $\tilde{\phi} \neq 1$  difficult to distinguish from the Schwarzschild solution  $\tilde{\phi} = 1$ . Note, however, that for  $\gamma \neq 1$ , the Einstein frame metric is distinct from the Schwarzschild metric (see (3.1)-(3.3)).

### A. Minimal radius, $r_c$ :

We wish to find the minimal radius  $r_c$  that a particle is allowed to reach, or equivalently, the minimum radius from which the particle is kinematically excluded, for the case where the singularity is kinematically forbidden ( $2\gamma < \Gamma$ ). To find  $r_c$  write (4.13) as

$$\xi = \left( \frac{r - r_0}{r + r_0} \right) \geq \left( \frac{1}{\mathcal{E}^2} \right)^{\frac{1}{|2\gamma - \Gamma|}} = \xi_c > 0; \quad (2\gamma - \Gamma) < 0 \quad (5.5)$$

where  $\Gamma > 2\gamma$  for the case where  $r = r_0$  is kinematically forbidden. This defines the kinematically excluded region with radius  $r \leq r_c$ . The constant parameter  $\xi_c = \xi_c(\mathcal{E})$  defined above is positive and decreases with increasing particle energy parameter  $\mathcal{E}$ . However, for any finite value of  $\mathcal{E}$  we have  $\xi > 0$ , so that the singularity is not reached. Setting  $r = r_c$  for  $\xi = \xi_c$ , we have

$$r_c = \left( \frac{1 + \xi_c}{1 - \xi_c} \right) r_0 \quad (5.6)$$

and, from (5.5),  $0 < \xi_c < 1$  since  $\xi_c \leq \xi < 1$ . The kinematically allowed region is that for which  $r > r_c$ . Restricting  $\xi_c$  to the range  $\xi_c \in (0, 1)$  implies that the particle energy parameter  $\mathcal{E} = \xi_c^{-\frac{|2\gamma - \Gamma|}{2}}$  for such a solution with an untouchable singularity is restricted to the range  $\mathcal{E} \in (1, \infty)$ . For a particle at rest ( $\mathcal{E} = 1$ ) at  $r \rightarrow \infty$  we see that  $r_c \rightarrow \infty$ , and the particle will not gravitationally fall inward. A particle that is projected inward with  $\mathcal{E} > 1$  will penetrate the dilaton cloud down to the radius  $r_c > r_0$ . A particle cannot exist at  $r < \infty$  with  $\mathcal{E} < 1$  without the application of an external force.

## B. Dilatonic acceleration

The corrected geodesic equation (4.1) may be rewritten as

$$\frac{du^\nu}{ds} = -\Gamma_{\alpha\beta}^\nu u^\alpha u^\beta + \partial_\mu(\ln m)(g^{\mu\nu} - u^\mu u^\nu) \quad (5.7)$$

The first term on the right hand side is the gravitational acceleration due to the metric field  $g_{\mu\nu}$ , while the second term on the right hand side represents the dilatonic acceleration, whose radial component is

$$a_r^{dil} = \partial_r(\ln m)[g^{rr} - (u^r)^2] = \frac{1}{2}\Gamma\partial_r(\ln \xi) \left[ \frac{\mathcal{E}^2 \xi^\Gamma}{|g_{rr}|g_{00}} \right] \quad (5.8)$$

where use has been made of (2.5) or (2.11), (3.4b), and (4.12). Noting that the term  $\partial_r(\ln \xi)$  is positive, as is the term in square brackets, we see that the sign of the radial component of the dilatonic acceleration  $a_r^{dil}$  depends upon the sign of  $\Gamma$ . For  $\Gamma = 0$  (the Schwarzschild solution)  $a_r^{dil} = 0$ . For  $\Gamma \neq 0$ ,

$$a_r^{dil} \text{ is } \begin{cases} > 0 \text{ for } \Gamma > 0 & (\text{radial outward acceleration}) \\ < 0 \text{ for } \Gamma < 0 & (\text{radial inward acceleration}) \end{cases} \quad (5.9)$$

This corresponds to the fact that  $m(r) \propto \tilde{\phi}^{-1/2} = \xi^{-\Gamma/2}$  decreases radially outward for  $\Gamma > 0$  and  $m(r)$  increases radially outward for  $\Gamma < 0$ , so that the constraint  $p_\mu p^\mu = m^2(r)$  implies that the test particle is attracted to regions of lower mass. The dilatonic repulsion ( $\Gamma > 0$ ) or dilatonic attraction ( $\Gamma < 0$ ) of a test particle is therefore sensitive to the parameter  $\Gamma$ .

## VI. SUMMARY

Static, spherically symmetric solutions of matter-free Brans-Dicke theory describe a class of naked singularities. The effect of the dilaton cloud (Brans-Dicke scalar field) on the radial motion of test particles has been found to have either a shielding or antishielding effect, depending on the values of the solution parameters. (Kaluza-Klein gravity has been examined as a special case.) The Brans-Dicke scalar field (see (3.4))  $\tilde{\phi} = \xi^\Gamma$  depends on the parameter  $\Gamma$ , which, in turn, depends upon the solution parameter  $\gamma \in [0, 1]$  and the Brans-Dicke parameter  $\omega_{BD}$ . The special case  $\gamma = 1$  yields the Schwarzschild solution, for which  $\Gamma = 0$  and there is no dilatonic effect, with  $\tilde{\phi} = 1$ . However, naked singularity solutions with  $\gamma \neq 1$  have  $\Gamma \neq 0$  with a radially varying dilaton field  $\tilde{\phi}(r)$ . In spite of this, since  $\Gamma$  depends inversely on  $\sqrt{\omega_{BD}}$ , a very large parameter  $\omega_{BD} \gg 1$  can give rise to  $\Gamma \approx 0$  with  $\gamma \neq 1$ . In this case the dilaton field is slowly varying with possibly negligible effects, though the Einstein frame metric of the spacetime depends on  $\gamma$  and deviates from the Schwarzschild case for  $\gamma \neq 1$ . Specifically, for  $\gamma \neq 1$ , but  $\Gamma \approx 0$ , the Brans-Dicke scalar is nearly frozen at a value  $\tilde{\phi} \approx 1$ , so that a test particle essentially follows a geodesic (see eqs.(5.7) and (5.8)), but not a Schwarzschild one.

A kinematical constraint on a test particle of mass  $m$  has been established, describing kinematically allowed regions where the particle may exist. This has been done in the Einstein frame where  $m(r) \propto \tilde{\phi}^{-1/2}(r)$ . In the Einstein frame we make use of the (extended) set of Xanthopoulos-Zannias solutions, for which the scalar field  $\phi$  can take positive or negative values, and hence  $\tilde{\phi}$ , which depends upon  $e^\phi$ , can be either an increasing or decreasing function of  $r$ . (In the Kaluza-Klein case this corresponds to the extra dimensional scale factor  $b(r)$  either vanishing or blowing up on the singularity.) The simple kinematical constraint obtained allows a determination of relative values of  $\gamma$  and  $\Gamma$  for which the naked singularity is kinematically accessible to the particle.

We find that the singularity is inaccessible, i.e., forbidden, for the parameters  $\Gamma > 2\gamma$ . In this case there can be no communication via massive particles between the singularity and a distant observer at infinity. For the case of  $\Gamma > 0$  there is a dilatonic repulsion of the test particle, and for  $\Gamma < 0$  there is a dilatonic attraction. The dilatonic acceleration is sensitive to the parameter  $\Gamma$ , and for the case of  $\Gamma > 2\gamma$  the closest distance of approach of the particle to the singularity is found to depend upon the particle's energy parameter  $\mathcal{E}$  and the quantity  $(\Gamma - 2\gamma)$ . For  $\Gamma \leq 2\gamma$  the particle will be gravitationally trapped if  $\mathcal{E} < 1$ , but can escape to  $r \rightarrow \infty$  if  $\mathcal{E} > 1$ . On the other hand, for  $\Gamma > 2\gamma$  the particle cannot be gravitationally trapped.

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